

# Existence of a unique positive solution for a singular fractional boundary value problem

Karimov E.T.<sup>\*†</sup> and Sadarangani K.<sup>‡</sup>

## Abstract

In the present work, we discuss the existence of a unique positive solution of a boundary value problem for nonlinear fractional order equation with singularity. Precisely, order of equation  $D_{0+}^{\alpha}u(t) = f(t, u(t))$  belongs to  $(3, 4]$  and  $f$  has a singularity at  $t = 0$  and as a boundary conditions we use  $u(0) = u(1) = u'(0) = u'(1) = 0$ . Using fixed point theorem, we prove the existence of unique positive solution of the considered problem.

**Keywords:** Nonlinear fractional differential equations, singular boundary value problem, positive solution.

*2000 AMS Classification:* MSC 34B16

## 1. Introduction

In this paper, we study the existence and uniqueness of positive solution for the following singular fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t)), & 0 < t < 1 \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \quad (1)$$

where  $\alpha \in (3, 4]$ , and  $D_{0+}^{\alpha}$  denotes the Riemann-Liouville fractional derivative. Moreover,  $f : (0, 1] \times [0, \infty) \rightarrow [0, \infty)$  with  $\lim_{t \rightarrow 0+} f(t, -) = \infty$  (i.e.  $f$  is singular at  $t = 0$ ).

Similar problem was investigated in [1], in case when  $\alpha \in (1, 2]$  and with boundary conditions  $u(0) = u(1) = 0$ . We note as well work [2], where the following problem

$$\begin{cases} D^{\alpha}u + f(t, u, u', D^{\mu}u) = 0, & 0 < t < 1 \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

was under consideration. Here  $\alpha \in (2, 3)$ ,  $\mu \in (0, 1)$  and function  $f(t, x, y, z)$  is singular at the value of 0 of its arguments  $x, y, z$ .

We would like notice some related recent works [3-5], which consider higher order fractional nonlinear equations for the subject of the existence of positive solutions.

---

<sup>\*</sup> Department of Mathematics and Statistics, Sultan Qaboos University, Al-Khoud 123, Muscat, Oman, Email: [erkinjon@gmail.com](mailto:erkinjon@gmail.com)

<sup>†</sup>Corresponding author

<sup>‡</sup> Departamento de Matematicas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain, Email: [ksadaran@dma.ulpgc.es](mailto:ksadaran@dma.ulpgc.es)

## 2. Preliminaries

We need the following lemma, which appear in [6].

**Lemma 1.** (Lemma 2.3 of [6]) Given  $h \in C[0, 1]$  and  $3 < \alpha \leq 4$ , a unique solution of

$$\begin{cases} D_{0+}^{\alpha} u(t) = h(t), & 0 < t < 1 \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \quad (1)$$

is

$$u(t) = \int_0^1 G(t, s) h(s) ds,$$

where

$$G(t, s) = \begin{cases} \frac{(t-s)^{\alpha-1} + (1-s)^{\alpha-2} t^{\alpha-2} [(s-t) + (\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \leq s \leq 1 \\ \frac{(1-s)^{\alpha-2} t^{\alpha-2} [(s-t) + (\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1 \end{cases}$$

**Lemma 2.** (Lemma 2.4 of [6]) The function  $G(t, s)$  appearing in Lemma 1 satisfies:

- (a)  $G(t, s) > 0$  for  $t, s \in (0, 1)$ ;
- (b)  $G(t, s)$  is continuous on  $[0, 1] \times [0, 1]$ .

For our study, we need a fixed point theorem. This theorem uses the following class of functions  $\mathfrak{F}$ .

By  $\mathfrak{F}$  we denote the class of functions  $\varphi : (0, \infty) \rightarrow \mathbb{R}$  satisfying the following conditions:

- (a)  $\varphi$  is strictly increasing;
- (b) For each sequence  $(t_n) \subset (0, \infty)$

$$\lim_{n \rightarrow \infty} t_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} \varphi(t_n) = -\infty;$$

- (c) There exists  $k \in (0, 1)$  such that  $\lim_{t \rightarrow 0^+} t^k \varphi(t) = 0$ .

Examples of functions belonging to  $\mathfrak{F}$  are  $\varphi(t) = -\frac{1}{\sqrt{t}}$ ,  $\varphi(t) = \ln t$ ,  $\varphi(t) = \ln t + t$ ,  $\varphi(t) = \ln(t^2 + t)$ .

The result about fixed point which we use is the following and it appears in [7]:

**Theorem 3.** Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  a mapping such that there exist  $\tau > 0$  and  $\varphi \in \mathfrak{F}$  satisfying for any  $x, y \in X$  with  $d(Tx, Ty) > 0$ ,

$$\tau + \varphi(d(Tx, Ty)) \leq \varphi(d(x, y)).$$

Then  $T$  has a unique fixed point.

## 3. Main result

Our starting point of this section is the following lemma.

**Lemma 4.** Let  $0 < \sigma < 1$ ,  $3 < \alpha < 4$  and  $F : (0, 1] \rightarrow \mathbb{R}$  is continuous function with  $\lim_{t \rightarrow 0^+} F(t) = \infty$ . Suppose that  $t^{\sigma} F(t)$  is a continuous function on  $[0, 1]$ . Then the function defined by

$$H(t) = \int_0^1 G(t, s) F(s) ds$$

is continuous on  $[0, 1]$ , where  $G(t, s)$  is the Green function appearing in Lemma 1.

*Proof:* We consider three cases:

1. **Case No 1.**  $t_0 = 0$ .

It is clear that  $H(0) = 0$ . Since  $t^\sigma F(t)$  is continuous on  $[0, 1]$ , we can find a constant  $M > 0$  such that

$$|t^\sigma F(t)| \leq M \text{ for any } t \in [0, 1].$$

Moreover, we have

$$\begin{aligned} |H(t) - H(0)| &= |H(t)| = \left| \int_0^1 G(t, s) F(s) ds \right| = \left| \int_0^1 G(t, s) s^{-\sigma} s^\sigma F(s) ds \right| = \\ &= \left| \int_0^t \frac{(t-s)^{\alpha-1} + (1-s)^{\alpha-2} t^{\alpha-2} [(s-t) + (\alpha-2)(1-t)s]}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds \right. \\ &\quad \left. + \int_t^1 \frac{(1-s)^{\alpha-2} t^{\alpha-2} [(s-t) + (\alpha-2)(1-t)s]}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds \right| = \\ &= \left| \int_0^1 \frac{(1-s)^{\alpha-2} t^{\alpha-2} [(s-t) + (\alpha-2)(1-t)s]}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds \right| \leq \\ &\leq \frac{Mt^{\alpha-2}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} |(s-t) + (\alpha-2)(1-t)s| s^{-\sigma} ds + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\sigma} ds \leq \\ &\leq \frac{M(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} s^{-\sigma} ds + \frac{Mt^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \left(1 - \frac{s}{t}\right)^{\alpha-1} s^{-\sigma} ds. \end{aligned}$$

Considering definition of Euler's beta-function, we derive

$$|H(t) - H(0)| \leq \frac{M(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha)} B(1-\sigma, \alpha-1) + \frac{Mt^{\alpha-\sigma}}{\Gamma(\alpha)} B(1-\sigma, \alpha).$$

From this we deduce that  $|H(t) - H(0)| \rightarrow 0$  when  $t \rightarrow 0$ .

This proves that  $H$  is continuous at  $t_0 = 0$ .

2. **Case No 2.**  $t_0 \in (0, 1)$ .

We take  $t_n \rightarrow t_0$  and we have to prove that  $H(t_n) \rightarrow H(t_0)$ . Without loss of generality, we consider  $t_n > t_0$ . Then, we have

$$\begin{aligned}
|H(t_n) - H(t_0)| &= \left| \int_0^{t_n} \frac{(t_n-s)^{\alpha-1} + (1-s)^{\alpha-2} t_n^{\alpha-2} [(s-t_n) + (\alpha-2)(1-t_n)s]}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds + \right. \\
&\quad + \int_0^{t_0} \frac{(1-s)^{\alpha-2} t_n^{\alpha-2} [(s-t_n) + (\alpha-2)(1-t_n)s]}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds - \\
&\quad - \int_0^{t_0} \frac{(t_0-s)^{\alpha-1} + (1-s)^{\alpha-2} t_0^{\alpha-2} [(s-t_0) + (\alpha-2)(1-t_0)s]}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds - \\
&\quad \left. - \int_{t_0}^{t_n} \frac{(1-s)^{\alpha-2} t_0^{\alpha-2} [(s-t_0) + (\alpha-2)(1-t_0)s]}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds \right| = \\
&= \left| \int_0^{t_n} \frac{(1-s)^{\alpha-2} t_n^{\alpha-2} [(s-t_n) + (\alpha-2)(1-t_n)s]}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds + \int_0^{t_n} \frac{(t_n-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds - \right. \\
&\quad - \int_0^{t_0} \frac{(1-s)^{\alpha-2} t_0^{\alpha-2} [(s-t_0) + (\alpha-2)(1-t_0)s]}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds - \int_0^{t_0} \frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds \left. - \right. \\
&\quad \left. - \int_{t_0}^{t_n} \frac{(1-s)^{\alpha-2} (t_n^{\alpha-2} - t_0^{\alpha-2}) [(s-t_n) + (\alpha-2)(1-t_n)s]}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds + \right. \\
&\quad + \int_0^{t_0} \frac{(1-s)^{\alpha-2} t_0^{\alpha-2} [(s-t_n) + (\alpha-2)(1-t_n)s] - [(s-t_0) + (\alpha-2)(1-t_0)s]}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds + \\
&\quad \left. + \int_0^{t_0} \frac{[(t_n-s)^{\alpha-1} - (t_0-s)^{\alpha-1}]}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds + \int_{t_0}^{t_n} \frac{(t_n-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^\sigma F(s) ds \right| \leq \\
&\leq \frac{M |t_n^{\alpha-2} - t_0^{\alpha-2}|}{\Gamma(\alpha)} (\alpha-1) \int_0^1 (1-s)^{\alpha-2} s^{-\sigma} ds + \\
&\quad + \frac{M t_0^{\alpha-2}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} |t_n - t_0| (\alpha-1) s^{-\sigma} ds + \\
&\quad + \frac{M}{\Gamma(\alpha)} \int_0^{t_0} |(t_n-s)^{\alpha-1} - (t_0-s)^{\alpha-1}| s^{-\sigma} ds + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t_n} (t_n-s)^{\alpha-1} s^{-\sigma} ds \leq \\
&\leq \frac{M}{\Gamma(\alpha)} (t_n^{\alpha-2} - t_0^{\alpha-2}) (\alpha-1) B(1-\sigma, \alpha-1) + \frac{M(t_n-t_0)}{\Gamma(\alpha)} (\alpha-1) B(1-\sigma, \alpha-1) + \\
&\quad + \frac{M}{\Gamma(\alpha)} I_n^1 + \frac{M}{\Gamma(\alpha)} I_n^2,
\end{aligned}$$

where

$$I_n^1 = \int_0^{t_0} [(t_n-s)^{\alpha-1} - (t_0-s)^{\alpha-1}] s^{-\sigma} ds, \quad I_n^2 = \int_{t_0}^{t_n} (t_n-s)^{\alpha-1} s^{-\sigma} ds.$$

In the sequel, we will prove that  $I_n^1 \rightarrow 0$  when  $n \rightarrow \infty$ . In fact, as

$$[(t_n-s)^{\alpha-1} - (t_0-s)^{\alpha-1}] s^{-\sigma} \leq [|t_n-s|^{\alpha-1} - |t_0-s|^{\alpha-1}] s^{-\sigma} \leq 2s^{-\sigma}$$

and  $\int_0^1 2s^{-\sigma} ds = \frac{2}{1-\sigma} < \infty$ . By Lebesgue's dominated convergence theorem  $I_n^1 \rightarrow 0$  when  $n \rightarrow \infty$ .

Now, we will prove that  $I_n^2 \rightarrow 0$  when  $n \rightarrow \infty$ . In fact, since

$$I_n^2 = \int_{t_0}^{t_n} (t_n-s)^{\alpha-1} s^{-\sigma} ds \leq \int_{t_0}^{t_n} s^{-\sigma} ds = \frac{1}{1-\sigma} (t_n^{1-\sigma} - t_0^{1-\sigma})$$

and as  $t_n \rightarrow t_0$ , we obtain the desired result.

Finally, taking into account above obtained estimates, we infer that  $|H(t_n) - H(t_0)| \rightarrow 0$  when  $n \rightarrow \infty$ .

3. **Case No 3.**  $t_0 = 1$ .

It is clear that  $H(1) = 0$  and following the same argument that in Case No 1, we can prove that continuity of  $H$  at  $t_0 = 1$ .

**Lemma 5.** Suppose that  $0 < \sigma < 1$ . Then there exists

$$N = \max_{0 \leq t \leq 1} \int_0^1 G(t, s) s^{-\sigma} ds.$$

**Proof:** Considering representation of the function  $G(t, s)$  and evaluations of Lemma 4, we derive

$$\begin{aligned} \int_0^1 G(t, s) s^{-\sigma} ds &= \frac{1}{\Gamma(\alpha)} [t^{\alpha-\sigma} B(1-\sigma, \alpha) - t^{\alpha-1} (B(1-\sigma, \alpha-1) + \\ &+ (\alpha-2)B(2-\sigma, \alpha-1)) + (\alpha-1)t^{\alpha-2} B(2-\sigma, \alpha-1)]. \end{aligned}$$

Taking

$$B(1-\sigma, \alpha) = \frac{\alpha-1}{\alpha-\sigma} B(1-\sigma, \alpha-1); \quad B(2-\sigma, \alpha-1) = \frac{1-\sigma}{\alpha-\sigma} B(1-\sigma, \alpha-1),$$

into account we infer

$$\begin{aligned} \int_0^1 G(t, s) s^{-\sigma} ds &= \\ &= \frac{B(1-\sigma, \alpha-1)}{\Gamma(\alpha)} \left[ \frac{\alpha-1}{\alpha-\sigma} t^{\alpha-\sigma} - \left( 1 + \frac{(\alpha-2)(1-\sigma)}{\alpha-\sigma} \right) t^{\alpha-1} + \frac{(\alpha-1)(1-\sigma)}{\alpha-\sigma} t^{\alpha-2} \right]. \end{aligned}$$

Denoting  $L(t) = \int_0^1 G(t, s) s^{-\sigma} ds$ , from the last equality one can easily derive that  $L(0) = 0$ ,  $L(1) = 0$ . Since  $G(t, s) \geq 0$ , then  $L(t) \geq 0$  and as  $L(t)$  is continuous on  $[0, 1]$ , it has a maximum. This proves Lemma 5.

**Theorem 6.** Let  $0 < \sigma < 1$ ,  $3 < \alpha \leq 4$ ,  $f : (0, 1] \times [0, \infty)$  be continuous and  $\lim_{t \rightarrow 0^+} f(t, \cdot) = \infty$ ,  $t^\sigma f(t, y)$  be continuous function on  $[0, 1] \times [0, \infty)$ . Assume that there exist constants  $0 < \lambda \leq \frac{1}{N}$ , and  $\tau > 0$  such that for  $x, y \in [0, \infty)$  and  $t \in [0, 1]$

$$t^\sigma |f(t, x) - f(t, y)| \leq \frac{\lambda |x - y|}{\left(1 + \tau \sqrt{|x - y|}\right)^2}.$$

Then Problem (1) has a unique non-negative solution.

**Proof:** Consider the cone  $P = \{u \in C[0, 1] : u \geq 0\}$ . Notice that  $P$  is a closed subset of  $C[0, 1]$  and therefore,  $(P, d)$  is a complete metric space where

$$d(x, y) = \sup \{|x(t) - y(t)| : t \in [0, 1]\} \text{ for } x, y \in P.$$

Now, for  $u \in P$  we define the operator  $T$  by

$$(Tu)(t) = \int_0^1 G(t, s) f(s, u(s)) ds = \int_0^1 G(t, s) s^{-\sigma} s^\sigma f(s, u(s)) ds.$$

In virtue of Lemma 4, for  $u \in P$ ,  $Tu \in C[0, 1]$  and, since  $G(t, s)$  and  $t^\sigma f(t, y)$  are non-negative functions,  $Tu \geq 0$  for  $u \in P$ . Therefore,  $T$  applies  $P$  into itself.

Next, we check that assumptions of Theorem 3 are satisfied. In fact, for  $u, v \in P$  with  $d(Tu, Tv) > 0$ , we have

$$\begin{aligned}
d(Tu, Tv) &= \max_{t \in [0, 1]} |(Tu)(t) - (Tv)(t)| = \\
&= \max_{t \in [0, 1]} \left| \int_0^1 G(t, s) s^{-\sigma} s^\sigma (f(s, u(s)) - f(s, v(s))) ds \right| \leq \\
&\leq \max_{t \in [0, 1]} \int_0^1 G(t, s) s^{-\sigma} s^\sigma |f(s, u(s)) - f(s, v(s))| ds \leq \\
&\leq \max_{t \in [0, 1]} \int_0^1 G(t, s) s^{-\sigma} \frac{\lambda |u(s) - v(s)|}{(1 + \tau \sqrt{|u(s) - v(s)|})^2} ds \leq \max_{t \in [0, 1]} \int_0^1 G(t, s) s^{-\sigma} \frac{\lambda d(u, v)}{(1 + \tau \sqrt{d(u, v)})^2} ds = \\
&= \frac{\lambda d(u, v)}{(1 + \tau \sqrt{d(u, v)})^2} \max_{t \in [0, 1]} \int_0^1 G(t, s) s^{-\sigma} ds = \frac{\lambda d(u, v)}{(1 + \tau \sqrt{d(u, v)})^2} N \leq \frac{d(u, v)}{(1 + \tau \sqrt{d(u, v)})^2},
\end{aligned}$$

where we have used that  $\lambda \leq \frac{1}{N}$  and the non-decreasing character of the function  $\beta(t) = \frac{t}{(1 + \tau \sqrt{t})^2}$ . Therefore,

$$d(Tu, Tv) \leq \frac{d(u, v)}{(1 + \tau \sqrt{d(u, v)})^2}.$$

This gives us

$$\sqrt{d(Tu, Tv)} \leq \frac{\sqrt{d(u, v)}}{1 + \tau \sqrt{d(u, v)}}$$

or

$$\tau - \frac{1}{\sqrt{d(Tu, Tv)}} \leq -\frac{1}{\sqrt{d(u, v)}}$$

and the contractivity condition of the Theorem 3 is satisfied with the function  $\varphi(t) = -\frac{1}{\sqrt{t}}$  which belongs to the class  $\mathfrak{F}$ .

Consequently, by Theorem 3, the operator  $T$  has a unique fixed point in  $P$ . This means that Problem (1) has a unique non-negative solution in  $C[0, 1]$ . This finishes the proof.

An interesting question from a practical point of view is that the solution of Problem (1) is positive. A sufficient condition for that solution is positive, appears in the following result:

**Theorem 7.** Let assumptions of Theorem 6 be valid. If the function  $t^\sigma f(t, y)$  is non-decreasing respect to the variable  $y$ , then the solution of Problem (1) given by Theorem 6 is positive.

**Proof:** In contrary case, we find  $t^* \in (0, 1)$  such that  $u(t^*) = 0$ . Since  $u(t)$  is a fixed point of the operator  $T$  (see Theorem 6) this means that

$$u(t) = \int_0^1 G(t, s) f(s, u(s)) ds \text{ for } 0 < t < 1.$$

Particularly,

$$0 = u(t^*) = \int_0^1 G(T^*, s) f(s, u(s)) ds.$$

Since that  $G$  and  $f$  are non-negative functions, we infer that

$$G(t^*, s) f(s, u(s)) = 0 \quad a.e. (s) \quad (2)$$

On the other hand, as  $\lim_{t \rightarrow 0^+} f(t, 0) = \infty$  for given  $M > 0$  there exists  $\delta > 0$  such that for  $s \in (0, \delta)$   $f(s, 0) > M$ . Since  $t^\sigma f(t, y)$  is increasing and  $u(t) \geq 0$ ,

$$s^\sigma f(s, u(s)) \geq s^\sigma f(s, 0) \geq s^\sigma M \quad \text{for } s \in (0, \delta)$$

and, therefore,  $f(s, u(s)) \geq M$  for  $s \in (0, \delta)$  and  $f(s, u(s)) \neq 0$  a.e. (s). But this is a contradiction since  $G(t^*, s)$  is a function of rational type in the variable  $s$  and, consequently,  $G(t^*, s) \neq 0$  a.e. (s). Therefore,  $u(t) > 0$  for  $t \in (0, 1)$ . This finishes the proof.

#### 4. Acknowledgement

The present work is partially supported by the project MTM-2013-44357-P.

#### References

- [1] J.Caballero, J.Harjani, K.Sadarangani. *Positive solutions for a class of singular fractional boundary value problems*. Computers and Mathematics with Applications. 62(2011), pp.1325-1332.
- [2] S.Staněk. *The existence of positive solutions of singular fractional boundary value problems*. Computers and Mathematics with Applications. 62(2011), pp.1379-1388.
- [3] Z.Bai, W.Sun. *Existence and multiplicity of positive solutions for singular fractional boundary problems*. Computers and Mathematics with Applications. 63(2012), pp.1369-1381.
- [4] J.Xu, Z.Weï, W.Dong. *Uniqueness of positive solutions for a class of fractional boundary value problems*. Applied Mathematics Letters. 25(2012), pp.590-593.
- [5] S.Zhang. *Positive solutions to singular boundary value problem for nonlinear fractional differential equation*. Computers and Mathematics with Applications. 59(2010), pp.1300-1309.
- [6] X.Xu, D.Jiang, C.Yuan. *Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation*. Nonlinear Analysis 71 (2009), pp. 4676-4688.
- [7] D.Wardowski. *Fixed points of a new type of contractive mappings in complete metric spaces*. Fixed point theory and applications (2012), 2012:14.